MATH 124B: HOMEWORK 4

Suggested due date: September 5th, 2016

(1) Show that the solution of the Poisson problem

$$\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

is given by

$$u(x_0) = \iint_{\partial D} h(x) \frac{\partial G(x, x_0)}{\partial n} dS + \iiint_D f(x) G(x, x_0) dV.$$

(2) Show that the Green's function solution of the Dirichlet problem in 2 dimensions

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } x^2 + y^2 < a^2 \\ u = h & \text{on } x^2 + y^2 = a^2. \end{cases}$$

Using the Green's function

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0| - \frac{1}{2\pi} \log \left(\frac{r_0}{a} |\mathbf{x} - \mathbf{x}_0^*| \right).$$

(3) Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

(4) Show that for $\mathbf{x}_0 = 0$ on the three dimensional sphere, the Green's function is given by

$$G(\mathbf{x}, 0) = -\frac{1}{4\pi \|\mathbf{x}\|} + \frac{1}{4\pi a}.$$

(5) Find the Green's function for the half-ball $D = \{x^2 + y^2 + z^2 < a^2, \mid z > 0\}$.

SOLUTION

(1) We can show the integral representation formula for u such that $\Delta u = f$, as

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \iint_{\partial D} -u(\mathbf{x}) \frac{\partial}{\partial n} \left(\frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} \right) + \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} \frac{\partial u}{\partial n} dS - \frac{1}{4\pi} \iiint_D \frac{f}{\|\mathbf{x} - \mathbf{x}_0\|} dV$$

Define the harmonic function $H := H + \frac{1}{4\pi \|\mathbf{x} - \mathbf{x}_0\|}$, apply this to Green's second identity, then add to the integral representation formula to obtain the solution.

(2) Let $\rho = \|\mathbf{x} - \mathbf{x}_0\|$ and $\rho^* = \|\mathbf{x} - \mathbf{x}_0^*\|$. We compute the normal derivative,

$$\nabla G = \frac{\nabla \rho}{2\pi\rho} - \frac{\nabla \rho^*}{2\pi\rho^*}$$

where $\nabla\rho = \frac{\mathbf{x}-\mathbf{x}_0}{\rho}$ and $\nabla\rho^* = \frac{\mathbf{x}-\mathbf{x}_0^*}{\rho^*}$. Since $n = \frac{\mathbf{x}}{a}$,

$$\begin{aligned}\frac{\partial G}{\partial n} &= \nabla G \cdot n \\ &= \frac{\mathbf{x} - \mathbf{x}_0}{2\pi\rho^2} \cdot \frac{\mathbf{x}}{a} - \frac{(\mathbf{x} - \mathbf{x}_0^*)}{2\pi(\rho^*)^2} \cdot \frac{\mathbf{x}}{a} \\ &= \frac{a^2 - \|\mathbf{x}_0\|^2}{2\pi a \rho^2}.\end{aligned}$$

where we used the fact that $\mathbf{x}_0^* = \frac{a^2 \mathbf{x}_0}{\|\mathbf{x}_0\|^2}$ and $\rho^* = \frac{a}{\|\mathbf{x}_0\|^2}\rho$ for $\|\mathbf{x}\| = a$. Inserting this into the Green's integral representation gives us exactly the Poisson formula for the disk in 2 dimensions.

(3) We need to assume either the domain is bounded or the function is bounded as $\|\mathbf{x}\| \to \infty$ for uniqueness.

(4) We start with the Green's function for the three dimensional sphere given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} + \frac{a}{4\pi\|\mathbf{x}_0\|\|\mathbf{x} - \mathbf{x}_0^*\|}.$$

We need to show that $\lim_{\mathbf{x}_0 \to 0} G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}, 0)$. The only issue we may have is the second term, hence we bound for both sides

$$|\|\mathbf{x}\|\|\mathbf{x}_0\| - a^2| \leq \|\mathbf{x}_0\|\|\mathbf{x} - \mathbf{x}_0^*\| \leq \|\mathbf{x}\|\|\mathbf{x}_0\| + a^2$$

By squeeze theorem, we have $\lim_{\mathbf{x}_0 \to 0} \|\mathbf{x}_0\|\|\mathbf{x} - \mathbf{x}_0^*\| = a^2$, which gives us the result.

(5) Define the reflection about the z axis given by

$$\tilde{\mathbf{x}} = \widetilde{(x, y, z)} = (x, y, -z).$$

By reflecting about the z-axis, we get a Green's function which satisfies the harmonic conditions and vanishes on the boundary, consisting of the upper half sphere and the $z = 0$ disk. That is

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} + \frac{a}{4\pi\|\mathbf{x}_0\|\|\mathbf{x} - \mathbf{x}_0^*\|} + \frac{1}{4\pi\|\mathbf{x} - \tilde{\mathbf{x}_0}\|} - \frac{a}{4\pi\|\tilde{\mathbf{x}_0}\|\|\mathbf{x} - \tilde{\mathbf{x}_0}\|}.$$